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Viscous laminar flow in a curved pipe of elliptical cross-section

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In this paper, the analysis on secondary flow in curved elliptic pipes of Topakoglu & Ebadian (1985) has been extended up to a point where the rate-of-flow expression is obtained for any value of flatness ratio of the elliptic cross-section. The analysis is based on the double expansion method of Topakoglu (1967). Therefore, no approximation is involved in any step other than the natural limitation of the finite number of calculated terms of the expansions. The obtained results are systematically plotted against the curvature of centreline of the curved pipe for different values of Reynolds number.

1. Introduction

For flows through curved pipes of elliptical cross-section, very few theoretical and experimental investigations are available (Berger, Talbot & Yao 1983). The relevant studies are by Thomas & Walters (1965), Srivastava (1980), Takami & Sudou (1984) and Ito (1969). In the analysis of the study by Thomas & Walters (1965), only the secondary flow is investigated by using the Dean's (1927, 1928) formulation in which the simplified forms of momentum and continuity equations have been used. In Srivastava's (1980) analysis, almost the entire solution is based on a numerical approach, in which no explicit expressions have been presented. Also, his results of the numerical calculations are presented without any detail for only six selected flatness ratio values (four for horizontally placed ellipse and two for vertically placed ellipse). However, these results do not agree with the physical fact that an increasing secondary flow (for sections with larger flatness ratios) requires a decreasing rate of flow. In Takami & Sudou (1984), the solution of the problem is referred to the result of Ito (1969) which is an approximation solution based on a boundary-layer approach. As a result, in Takami & Sudou (1984) even the independent pertinent parameters of the problem could not have been found.

Another recent study for flow in curved elliptic pipes is due to Topakoglu & Ebadian (1985) in which only the secondary flow is investigated in a proper and systematic manner.

In this present paper, the analysis of Topakoglu & Ebadian (1985) has been extended up to a point where the rate of flow expression is obtained for any value of flatness ratio of the elliptic cross-section. The analysis is based on the double expansion method of Topakoglu (1967). Therefore, no approximation is involved in any step other than the natural limitation of the finite number of calculated terms of the expansions. The obtained results are systematically plotted against the curvature of centreline of the curved pipe for different values of Reynolds number.

2. Governing equations for primary and secondary flows

The curved-pipe geometry is shown in figure 1, where OX_3 represents the axis of symmetry of the curved pipe. This figure also shows an arbitrary cross-section of the elliptic pipe in a meridianal plane with an azimuth angle θ relative to the fixed coordinate axes $OX_1X_2X_3$.

The elliptic cross-section considered here has a semi-major axis A and a semi-minor axis B. In addition, the section is oriented in such a way that the major axis is along the direction of curvature of the curved pipe, and the minor axis is along the direction of the axis of symmetry of the curved pipe.

The dimensionless primary flow-velocity component w, and the dimensionless stream function ϕ, satisfy the following equations (cf. Topakoglu 1967):

$$\left.\begin{aligned} \nabla^2 w &= \frac{1}{(y+\sigma)^2} w(1-\sigma\phi_x) + \frac{1}{y+\sigma}\left[\sigma\frac{\partial(\phi,w)}{\partial(y,x)} - w_y - \sigma k_0\right], \\ \nabla^4\phi &= \frac{1}{\sigma}(w^2)_x + \frac{1}{y+\sigma}\left[2\nabla^2\phi_y + \sigma\frac{\partial(\phi,\nabla^2\phi)}{\partial(y,x)}\right] \\ &\quad - \frac{1}{(y+\sigma)^2}\left[3\phi_{yy} + \sigma\frac{\partial(\phi,\phi_y)}{\partial(y,x)} - 2\sigma\phi_x\nabla^2\phi\right] \\ &\quad + 3\frac{1}{(y+\sigma)^3}\phi_y(1-\sigma\phi_x), \\ \text{where}\quad \nabla^2 &\equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}. \end{aligned}\right\} \tag{2.1}$$

Subscripts x and y indicate partial differentiation with respect to these variables respectively.

3. Expansions of primary and secondary flows in terms of curvature

The first terms of the expansions of the primary and secondary flows, expressed in elliptic coordinates, respectively are (cf. Topakoglu & Ebadian 1985)

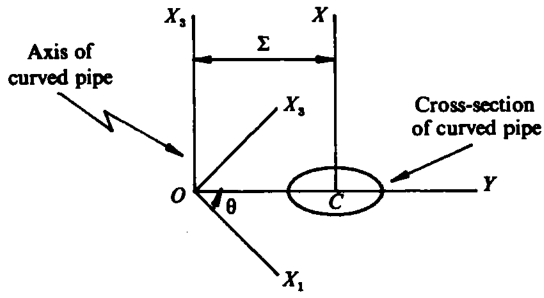
$$w_0 = Re\,(w_{00} - w_{02}\cos 2\eta), \tag{3.1}$$

where
$$w_{00} = (1-\xi^2)\left(1-\frac{m^4}{\xi^2}\right), \quad w_{02} = \frac{2m^2}{1+m^4}w_{00},$$

and

$$\phi_1 = -Re^2\left(\frac{1+m^2}{1+m^4}\right)^2\frac{1}{1-m^2}(F_1\sin\eta - m^2F_3\sin 3\eta + m^4F_5\sin 5\eta - m^6F_7\sin 7\eta), \tag{3.2}$$

where the expressions of the functions F_1–F_7 for horizontally and vertically placed elliptic peripheries are given in Topakoglu & Ebadian (1985). Nevertheless, in the case of horizontally placed elliptic peripheries, simpler forms of these expressions are developed and are given in the Appendix.



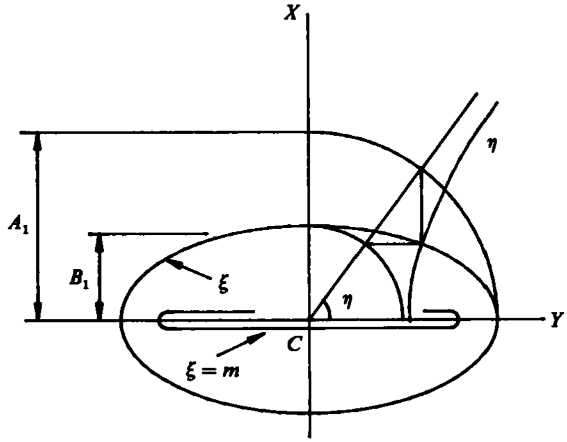


FIGURE 1. Definition of coordinates.

4. Second term of the primary flow

The second term of the primary flow satisfies the following Poisson's equation:

$$\nabla^2 w_1 = k_0 y - w_{0y} + \frac{\partial(\phi, w_0)}{\partial(y, x)}. \tag{4.1}$$

The terms in the above equation can be expressed in terms of the elliptic coordinates ξ and η, defined by the relations

$$y = \frac{1}{1+m^2}\left(\xi + \frac{m^2}{\xi}\right)\cos\eta, \quad x = \frac{1}{1+m^2}\left(\xi - \frac{m^2}{\xi}\right)\sin\eta. \tag{4.2}$$

The solution of (4.1), after the above transformation and by the proper boundary condition (Topakoglu & Ebadian 1985) is obtained as

$$\begin{aligned} w_1 = {} & Re\,(w_{11}\cos\eta - m^2 w_{13}\cos 3\eta) \\ & + Re^3\,(w_{31}\cos\eta - m^2 w_{33}\cos 3\eta + m^4 w_{35}\cos 5\eta - m^6 w_{37}\cos 7\eta + m^8 \omega_{39}\, 9\eta), \end{aligned} \tag{4.3}$$

where the coefficients w_{ij} are algebraic functions of ξ and m. Each of these functions is obtained explicitly in terms of ξ and m but they are not included here because their forms are too complicated. However, they will affect the rate-of-flow expression.

It must be noted that in the special case of $m = 0$, the expression for w_1 reduces to the corresponding expression for a circular curved pipe (Topakoglu 1967).

5. Second term of the secondary flow

The second term of the secondary flow satisfies the following non-homogeneous biharmonic equation:

$$\nabla^4\phi_2 = 2(w_0 w_1)_x + 2\nabla^2\phi_{1y} + \frac{\partial(\phi, \nabla^2\phi_1)}{\partial(y,x)}. \tag{5.1}$$

After transforming each term into elliptic coordinates ξ and η, the solution of (5.1) under the proper boundary conditions (Topakoglu & Ebadian 1985) is obtained as

$$\begin{aligned}\phi_2 = {} & Re^2\,(g_{22}\sin 2\eta - m^2 g_{24}\sin 4\eta + m^4 g_{26}\sin 6\eta - m^6 g_{28}\sin 8\eta) \\ & + Re^4\,(g_{42}\sin 2\eta - m^2 g_{44}\sin 4\eta + m^4 g_{46}\sin 6\eta - m^6 g_{48}\sin 8\eta \\ & + m^8 g_{410}\sin 10\eta - m^{10} g_{412}\sin 12\eta + m^{12} g_{414}\sin 14\eta),\end{aligned} \tag{5.2}$$

where the coefficients g_{ij} and g_{ijk} are algebraic functions of ξ and m. Each of these functions is obtained explicitly in terms of ξ and m, but their forms are too complicated to be given here. However, they will affect the rate-of-flow expression.

In the special case of $m = 0$, it is verified that the expression for ϕ_2 reduces to the corresponding expression for a circular curved pipe (Topakoglu 1967).

6. Third term of the primary flow

The third term of the primary flow satisfies the following Poisson's equation:

$$\nabla^2 w_2 = w_0(1-\phi_{1x}) + \frac{\partial(\phi_2, w_0)}{\partial(y,x)} + \frac{\partial(\phi_1, w_1)}{\partial(y,x)} - \frac{\partial(\phi_1, w_0)}{\partial(y,x)}y - w_{1y} + w_{0y}y - k_0 y^2. \tag{6.1}$$

After transforming each term into a new form in terms of elliptic coordinates ξ and η, the solution of (6.1) under the proper boundary conditions (Topakoglu & Ebadian 1985) is obtained as

$$\begin{aligned}w_2 = {} & Re\,(w_{10} - w_{12}\cos 2\eta + m^2 w_{14}\cos 4\eta) \\ & + Re^3\,(w_{30} - w_{32}\cos 2\eta + m^2 w_{34}\cos 4\eta - m^4 w_{36}\cos 6\eta \\ & + m^6 w_{38}\cos 8\eta - m^8 w_{310}\cos 10\eta) \\ & + Re^5\,(w_{50} - w_{52}\cos 2\eta + m^2 w_{54}\cos 4\eta - m^4 w_{56}\cos 6\eta \\ & + m^6 w_{58}\cos 8\eta - m^8 w_{510}\cos 10\eta \\ & + m^{10} w_{512}\cos 12\eta - m^{12} w_{514}\cos 14\eta \\ & + m^{14} w_{516}\cos 16\eta),\end{aligned} \tag{6.2}$$

where the coefficients w_{ij} and w_{ijk} are algebraic functions of ξ and m. Each of these functions is obtained explicitly in terms of ξ and m, but their forms are too complicated to be given here. However, they will affect the rate-of-flow expression.

In the special case of $m = 0$, it is verified that the expression for w_2 becomes equal to the corresponding expression for a circular curved pipe (Topakoglu 1967).

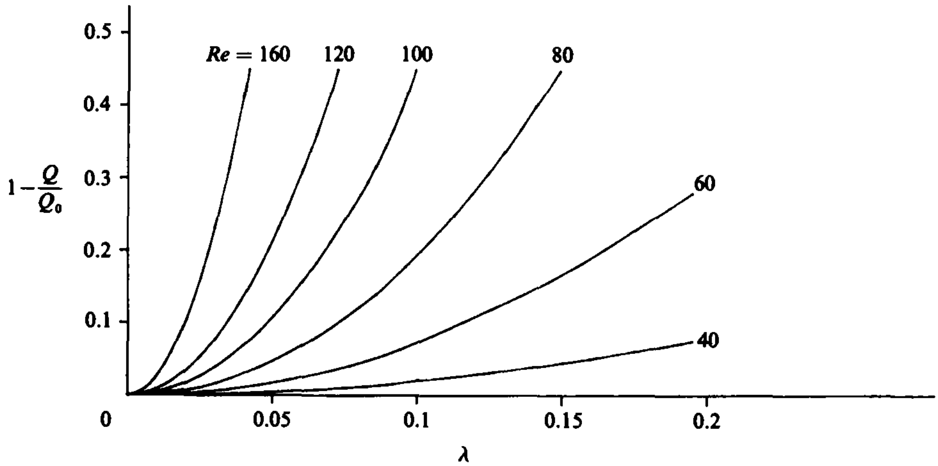


FIGURE 2. Rate of flow reduction ($m = 0$; $c_1 = 1.00$).

7. Rate of flow

The mass rate of flow Q through a curved pipe with an elliptic cross-section is

$$Q = \rho \int_{\eta=0}^{\eta=2\pi} \int_{\xi=m}^{\xi=1} W \, \mathrm{d}S, \tag{7.1}$$

where ρ is the density and $\mathrm{d}S$ is the element of area expressed in elliptic coordinates as

$$\mathrm{d}S = A^2\left(1+\frac{m^4}{\xi^4}-2\frac{m^2}{\xi^2}\cos 2\eta\right)\xi \, \mathrm{d}\xi \, \mathrm{d}\eta. \tag{7.2}$$

After substituting $\mathrm{d}S$ and W by using (3.1), (4.3), (6.2), one obtains

$$Q = 2\pi\rho A v Re\,\{q_{01}+\lambda^2[q_{21}+Re^2 q_{23}+Re^4 q_{25}]+\text{higher-order terms}\}, \tag{7.3}$$

where the factors q_{01}, q_{21}, q_{23} and q_{25} are functions of m. Their expressions have been obtained explicitly in terms of m. They are not given here owing to their complexities. However, the resulting curves presented in this paper are based on these functions.

The ratio of rate of flow in a curved pipe of elliptic cross-section to that of a straight pipe having the same cross-section and the same pressure gradient as that measured along the centreline of the curved pipe, from (7.3), is

$$\frac{Q}{Q_0} = 1+\lambda^2\left[\frac{q_{21}}{q_{01}}+Re^2\frac{q_{23}}{q_{01}}+Re^4\frac{q_{25}}{q_{01}}\right]. \tag{7.4}$$

The above expression, (7.4), involves only the most significant contribution of the effect of the curvature. The effect of the other terms can be obtained when further terms of the expansions are calculated. It is seen that the rate-of-flow ratio, besides depending on the shape of the section (the ellipticity factor m) and the curvature, also depends on the Reynolds number Re. This dependence will be presented in the next section.

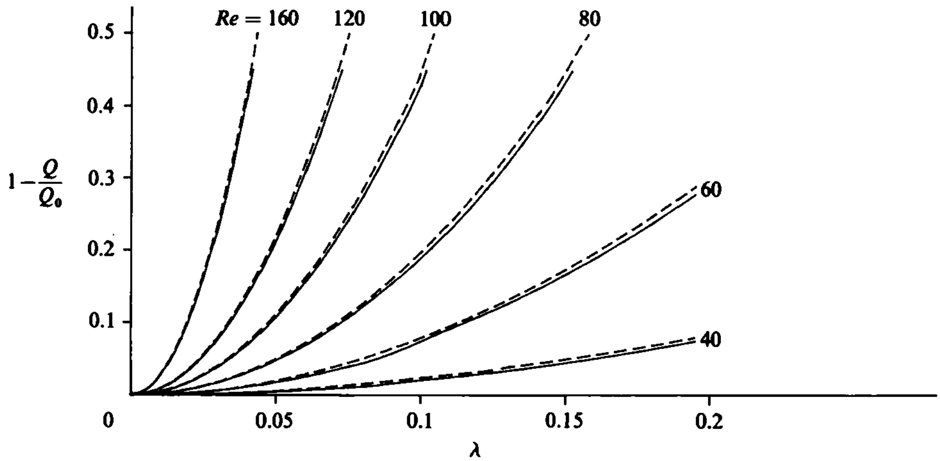


FIGURE 3. Rate of flow reduction ($m = 0.07$; $c_1 = 0.98$). The dotted curves refer to Srivastava (1980).

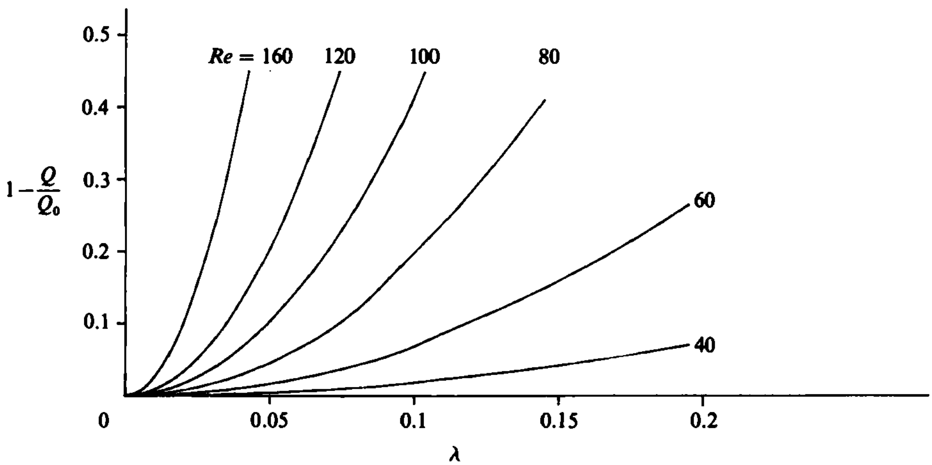


FIGURE 4. Rate of flow reduction ($m = 0.16$; $c_1 = 0.90$).

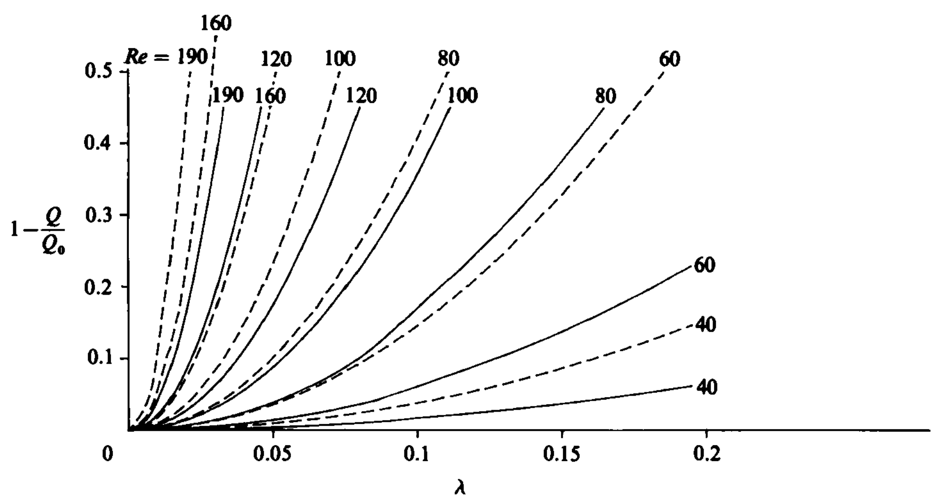


FIGURE 5. Rate of flow reduction ($m = 0.27$; $c_1 = 0.75$). The dotted curves refer to Srivastava (1980).

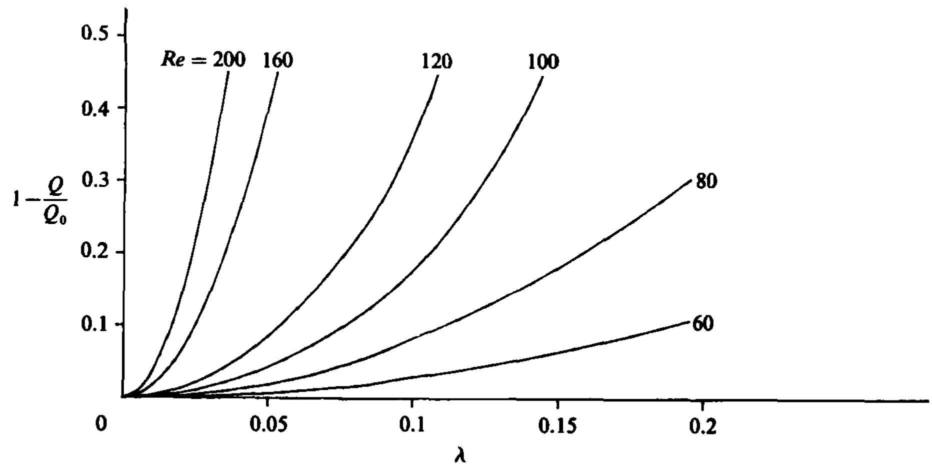


FIGURE 6. Rate of flow reduction ($m = 0.30$; $c_1 = 0.70$).

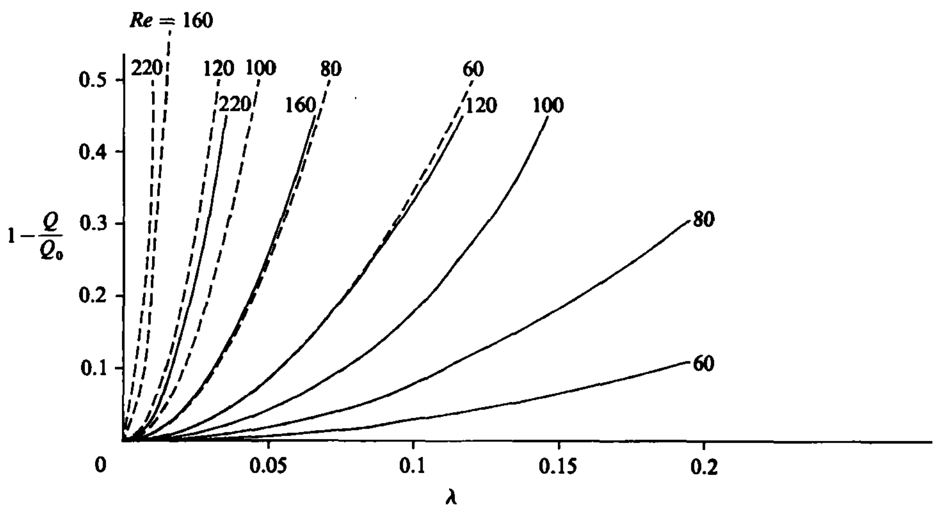


FIGURE 7. Rate of flow reduction ($m = 0.41$; $c_1 = 0.50$). The dotted curves refer to Srivastava (1980).

8. Results and discussion

Using the obtained results and some selected values for the flatness ratios of the periphery of the cross-section, the rate-of-flow reduction $(1-Q/Q_0)$ is plotted systematically against the curvature of the elliptic pipe for different values of the Reynolds number. The five flatness ratios are selected as $c_1 = 1.0, 0.9, 0.7, 0.5, 0.3$. In order to compare the results directly with the findings of Srivastava (1980), the flatness ratio of the ellipse is defined as the square of the ratio of the minor to the major radius of the elliptic periphery ($c_1 = B^2/A^2$). Furthermore, three more values of c_1 (0.98, 0.75 and 0.25) are plotted to compare the results of Srivastava (1980) and the present paper.

The rate-of-flow-reduction curves corresponding to the eight values of c_1 are presented in figures 2–9, successively. In each case, the Reynolds numbers are selected as such that a fairly wide coverage is obtained.

In general, a positive reduction ratio is observed in each case. However, it must

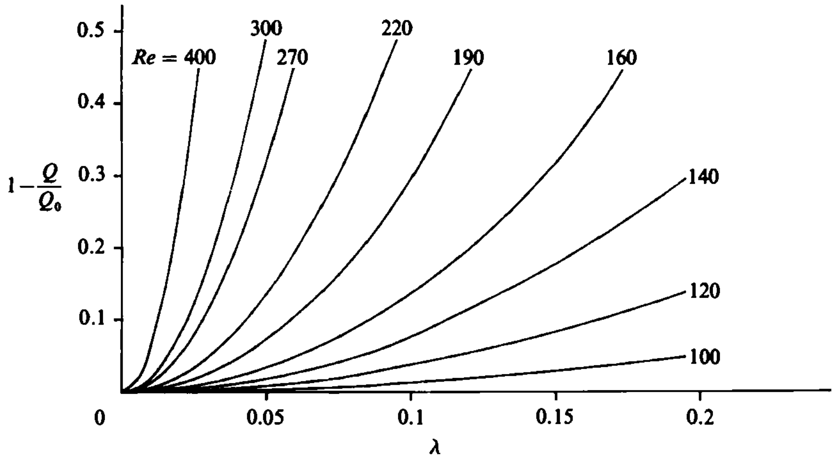


FIGURE 8. Rate of flow reduction ($m = 0.54$; $c_1 = 0.30$).

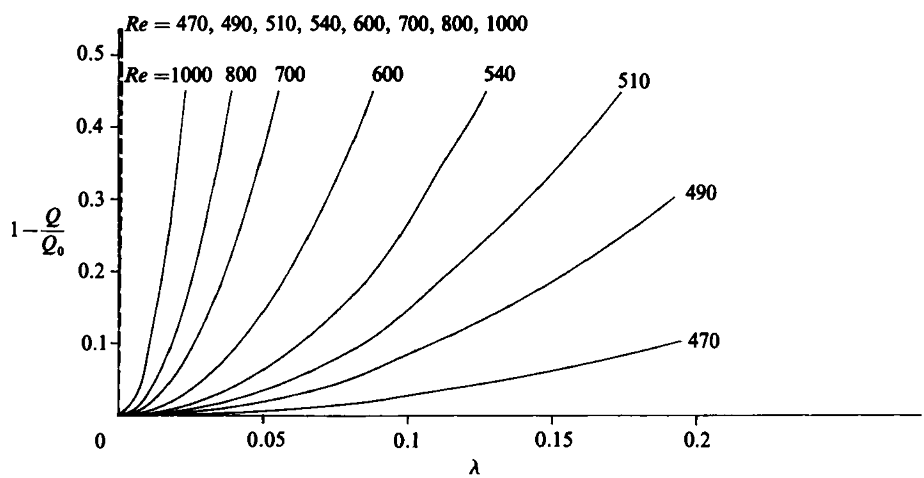


FIGURE 9. Rate of flow reduction ($m = 0.57$; $c_1 = 0.25$). The dotted curves, which are combined into a single line, refer to Srivastava (1980).

be noted that the presented results reflect only the effect of the most significant term of the curvature on the rate of flow. The effect of the other terms can only be seen if further terms of the expansions are calculated.

The first case of $c_1 = 1.0$, which is the circular cross-section, is included in order to be able to see the gradual transition from circular pipe to an elliptic pipe. For comparison, the findings of Srivastava (1980) are also plotted as dotted curves for the cases of $c_1 = 0.98$, 0.75, 0.50 and 0.25.

When figures 2–9 are compared to each other, one can see that for a fixed value of the Reynolds number and at fixed curvature, as the flatness ratio decreases, the rate of flow reduction decreases as well. This is because a smaller flatness ratio means a smaller secondary flow, which also means a smaller rate of flow reduction.

On the other hand, the results of Srivastava (figures 3, 5, 7 and 9) do not agree with the above fact. In his case, for a fixed Reynolds number and a fixed curvature, a smaller flatness ratio results in a higher rate of flow reduction. This discrepancy is especially noticeable for $c_1 = 0.25$ in figure 9 in which all dotted lines are combined into a single line, and the above stated fact is totally unobservable.

In the literature no experimental result using the pertinent parameters for the flow in a curved pipe has been reported. Consequently, correlation of analytical studies with experiments is not possible.

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Appendix

The functions involved in (4.6) are

$$F_1 = \tfrac{1}{288}(A^{(1)}e_2 - B^{(1)}e_6\,\xi^2 + c_{10}\,e_{10}\,\xi^4 - c_{14}\,e_{14}\,\xi^6)\,\xi,$$
$$F_3 = \tfrac{1}{480}(\tfrac{5}{3}B^{(1)}e_2 - A^{(3)}e_6\,\xi^2 + B^{(3)}e_{10}\,\xi^4 - 3b_1\,e_{14}\,\xi^6)\,\xi,$$
$$F_5 = \tfrac{1}{288}(c_{10}\,e_2 - \tfrac{3}{5}B^{(3)}e_6\,\xi^2 + A^{(5)}e_{10}\,\xi^4 - B^{(5)}e_{14}\,\xi^6)\,\xi,$$
$$F_7 = \tfrac{1}{1440}(5c_{14}\,e_2 - 9b_1\,e_6\,\xi^2 + 5B^{(5)}e_{10}\,\xi^4 - B^{(7)}e_{14}\,\xi^6)\,\xi,$$

where
$$e_2 = 1-\frac{m^2}{\xi^2},\quad e_6 = 1-\frac{m^6}{\xi^6},\quad e_{10} = 1-\frac{m^{10}}{\xi^{10}},\quad e_{14} = 1-\frac{m^{14}}{\xi^{14}},$$
$$c_{10} = 6(1-m^{10}) - 2m^2(1-m^6) - 3m^4(1-m^2),$$
$$c_{14} = 1-m^{16},\quad b_1 = 1-m^2,$$
$$A^{(1)} = \frac{1}{u_1}(c_{10}\,u_4 - c_{14}\,u_5),\quad B^{(1)} = \frac{1}{u_1}(c_{10}\,u_2 - c_{14}\,u_3),$$
$$A^{(3)} = \frac{1}{u_4}(\tfrac{5}{3}B^{(1)}u_2 + 3b_1\,u_6),\quad B^{(3)} = \frac{1}{u_4}(\tfrac{5}{3}B^{(1)}u_1 + 3b_1\,u_5),$$
$$A^{(5)} = \frac{1}{u_6}(\tfrac{3}{5}B^{(3)}u_5 - c_{10}\,u_3),\quad B^{(5)} = \frac{1}{u_6}(\tfrac{3}{5}B^{(3)}u_4 - c_{10}\,u_2),$$
$$B^{(7)} = \frac{1}{u_6}(5c_{14}\,u_2 - 9b_1\,u_4),$$
$$u_1 = 3(1-m^2)(1+m^6) - (1+m^2)(1-m^6),$$
$$u_2 = 5(1-m^2)(1+m^{10}) - (1+m^2)(1-m^{10}),$$
$$u_3 = 7(1-m^2)(1+m^{14}) - (1+m^2)(1-m^{14}),$$
$$u_4 = 5(1-m^6)(1+m^{10}) - 3(1+m^6)(1-m^{10}),$$
$$u_5 = 7(1-m^6)(1+m^{14}) - 3(1+m^6)(1-m^{14}).$$

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